

8. The essence of mathematics

1. We can now proceed, on the basis of all the above, to general conclusions about the essence of mathematics.

The essence of mathematics was expressed by Engels in one of the sections of *Anti-Dühring*, and we will cite here this remarkable passage.

In Engels' formulations the reader will easily recognize what was said above, for example, about arithmetic and geometry; and this is understandable: we set forth the actual history of the emergence and development of mathematics, guided in its understanding by dialectical materialism. Dialectical materialism leads to correct conclusions precisely because it does not impose anything on facts, but considers facts as they are, that is, in their necessary relations and development.

Engels begins his presentation of the essence of mathematics with critical remarks on the silliness of Dühring's views, in particular on the false view that mathematics deals with the creations of "pure reason" independently of experience. Engels writes:

"But it is not true that in pure mathematics reason deals only with the products of its own creativity and imagination. The concepts of number and figure are not taken from anywhere, but only from the real world. The ten fingers on which people learned to count, i.e., to perform the first arithmetic operation, represent anything but the product of the mind's own free creativity. In order to count, one must have not only objects subject to counting, but already possess the ability to abstract from all other properties of these objects except number, and this ability is the result of a long historical development based on experience. Both the concept

of number and the concept of figure are borrowed exclusively from the outside world and did not arise in the mind from pure thinking. There had to be things that had a certain shape, and these shapes had to be compared before one could reach the concept of a figure. Pure mathematics has as its object the spatial forms and quantitative relations of the real world, therefore a very real material. The fact that this material takes an extremely abstract form can only weakly obscure its origin from the outside world. But in order to be able to investigate these forms and relations in their pure form, it is necessary to separate them completely from their content, to leave the latter aside as something irrelevant; in this way we obtain points devoid of measurements, lines devoid of thickness and width, different a and b , x and y , constant and variable values, and only in the very end we come to products of free creativity and imagination of reason itself, namely - to imaginary values. Similarly, the seemingly a priori deduction of mathematical quantities from each other does not prove their a priori origin, but only their rational reciprocal relationship. Before arriving at the idea of deducing the *shape* of a cylinder from the rotation of a rectangle around one of its sides, it was necessary to investigate some number of real rectangles and cylinders, albeit in very imperfect forms. Like all other sciences, mathematics arose from the *practical needs* of men: from measuring the areas of land and the capacity of vessels, from calculating time, and from mechanics. But as in all other fields of thought, laws which are abstracted from the real world, at a certain stage of development, are detached from the real world, are opposed to it as something independent, as laws which appeared from without, with which the world must conform. So it was with society and the state; in this way, not otherwise, *pure* mathematics is *applied* later to the world, although it is borrowed from this very world

and only expresses a part of its inherent forms of connections, - and *only therefore* can it be applied at all”¹.

2. Thus, Engels emphasizes that mathematics reflects reality, that it arose from the practical needs of people and that the emergence of its first concepts and provisions was the result of a long, experience-based historical development. We have already traced this in some detail on the examples of arithmetic and geometry.

We made sure, in particular, that this is how the notions of number, magnitude, geometric figure appeared and that they reflect real quantitative relations and spatial forms of reality. Similarly, the basic concepts of analysis reflect real quantitative relations, they were formed gradually on the basis of generalization of huge concrete material; so, the concept of function reflects in a generalized, abstract form drastic dependence between real quantities.

Summing up all this, Engels comes to the main conclusion that *mathematics has as its subject some quite real material, but considers it in complete abstraction from concrete content and qualitative features*. In this way, as we see, mathematics differs from the natural sciences, and Engels clearly separates it from natural science².

The possibility of such an abstract consideration of the subject of mathematics has an objective basis in the subject itself. Those general forms, relations, interrelations and laws which are reflected in mathematics, independent of qualitative features or concrete content, exist objectively, independently of our consciousness. Only the existence of number as an

¹Engels, Anti-Dühring

²Engels, Anti-Dühring

objective property of a set of objects, the independence of relations between numbers from qualitative features of objects, the richness of these relations made arithmetic possible. Where there are no such general forms and relations, indifferent to the content, mathematical consideration is also impossible.

3. This basic feature of mathematics determines its other characteristic features. In §2 we considered some of them specifically on the example of arithmetic. These are the specific “language of formulas”, the breadth of applications, the abstract nature of mathematical conclusions, their logical inevitability and persuasiveness. This speculative character of mathematics is a very essential feature of it, and we will examine this feature in more detail.

If we have abstracted, say, the concept of number from its concrete foundations and consider whole numbers in general, outside any relation to particular aggregations of objects, it is clear that we cannot make experiments on such abstract numbers. Staying at this level of abstraction and not returning to concrete objects, we can get new conclusions about numbers only by reasoning from the very notion of number. The same applies, of course, to all other mathematical conclusions. Staying within the limits of pure geometry, i.e. considering geometrical figures in full abstraction from any qualitative, concrete content, we cannot obtain new conclusions otherwise than by reasoning proceeding from the very concept of this or that figure, from the very basic concepts or axioms of geometry. Thus, the properties of a circle are deduced from the notion of the circle as a geometric place of points equidistant from a given point, without thinking about the verification of each theorem by experience.

The abstract nature of mathematics already predetermines the fact that mathematical theorems are proved only by reasoning from the concepts themselves.

It can be said that in mathematics one investigates quantitative relations, having in mind only what is contained in their very definition. Accordingly, mathematical conclusions are obtained by reasoning from definitions. Of course, it would be wrong to understand these words too literally and to assume that sufficiently rigorous definitions of mathematical concepts were actually formulated before the corresponding mathematical theory was created; in fact, the concepts themselves were refined along with the development of the theory, as a result of its development. An in-depth analysis of the concept of the whole number, as well as the exact formulation of the axioms of geometry, were given not in antiquity, but by the end of the 19th century. Any concept, no matter how precisely defined it may seem, is still mobile, it develops and refines with the development of science. This is quite proved by the development of mathematics with respect to all its concepts and this confirms the main statement of dialectics that there is nothing in the world that would be absolutely immovable and would not develop in any way. Therefore with respect to mathematical notions we can speak, firstly, only about their sufficient but by no means perfect definiteness, and secondly, it is necessary to keep in mind that the accuracy and clarity of their definition, the depth of their analysis develop with the development of mathematics. We shall have an occasion to consider this flexibility of mathematical notions in the next section, but now with this point in mind we would like to focus on their sufficient definiteness.

It is this certainty of mathematical concepts, together with the general validity of logic itself, that gives rise to the intrinsic persuasiveness and logical necessity of mathematics's conclusions. The inevitability of the

speculative conclusions of mathematics gives rise to the erroneous notion that mathematics has its basis in pure thought, that it is a priori and not derived from experience, that it does not reflect reality. This was the view held, for example, by the famous German philosopher Kant. This deeply erroneous idealistic view stems in part from the fact that mathematics is viewed not in its real emergence and development, but in its ready-made form. But this approach is fundamentally unsustainable for the simple reason that it does not correspond to the actual state of affairs. That mathematics is not a priori, but arose from experience, is a firmly established fact. Incidentally, Eudemus of Rhodes, whom we quoted in §3, wrote about the actual emergence of geometry.

Not only the very concepts of mathematics, but also its conclusions, its methods reflect reality. This important circumstance is exactly what Engels reveals when he writes that “the deduction of mathematical quantities from one another, seemingly a priori, proves not their a priori origin, but only their rational mutual connection.” Mathematical deductions and proofs arose as a reflection of real connections that people explored through experience. The addition of numbers reflects the real connection of several sets of objects into one. The well-known proof of the theorem of the equality of triangles, which speaks of their superposition, undoubtedly has as its source the operation of actually attaching objects to each other, which is constantly performed when comparing their sizes. The calculation of volumes by integration reflects in an abstract form the actual possibility of forming bodies out of thin layers or cutting them into such layers. More complex mathematical proofs are the result of further development proceeding from such material foundations.

4. The complete abstraction of the subject of mathematics from all concrete-

ness and the speculative character of mathematical conclusions based on this entails another important feature of mathematics: mathematics investigates not only quantitative relations and spatial forms that are directly abstracted from reality, but also such relations and forms that are defined within mathematics itself on the basis of already established mathematical notions and theories. It is this peculiarity of mathematics that Engels draws attention to when, pointing to the emergence of the concepts of point, line, constant and variable magnitude, he says: "... only at the very end do we reach the products of the free creativity and imagination of reason itself, namely, imaginary values".

It is a historical fact that imaginary numbers were not taken from reality in the same sense as, say, integers. They appeared originally within mathematics itself, out of the necessary development of algebra, as roots of equations of the form $x^2 = -a$ (where $a > 0$). Although they gradually began to be handled quite freely, their real meaning remained unclear for a long time, which is why they were called "imaginary". Then their geometrical interpretation was discovered and they found many important applications. Similarly, Lobachevsky's geometry arose as a product of the work of this great scientist; he had not yet seen its real meaning and therefore called it "imaginary geometry". But it was not a free play of the mind, but an inevitable conclusion from the basic concepts of geometry, and Lobachevsky regarded it as a possible theory of spatial forms and relations. Therefore, the "free creativity and imagination" of which Engels speaks cannot be understood as mere arbitrary thought. Free creativity in science is a conscious logical necessity determined by initial, taken from experience concepts and positions.

At the new stage of development of mathematics, which began with the

construction of Lobachevsky's geometry and the exact theory of imaginary numbers, new concepts and theories continually emerge, founded on already established concepts and theories without borrowing them directly from reality. Mathematics defines and investigates possible forms of reality, which is precisely one of the decisive features of the latest stage of its development.

The theory of dialectical materialism provides a correct understanding of this peculiarity. Lenin wrote: "Cognition is man's reflection of nature. But it is not a simple, not direct, not a complete reflection, but a process of a number of abstractions, formation, establishment of concepts, laws. . . ." ³. Metaphysical materialism also recognizes cognition, particularly mathematics, as a reflection of nature. However, as Lenin notes, the trouble with metaphysical materialism is its inability to apply dialectics to the theory of reflection ⁴. Metaphysical materialism does not understand the complexity of this reflection, does not understand that it goes through a series of abstractions, by forming new concepts, by constructing new theories on the basis of already established concepts and theories, by considering not only the given in experience, but also the possible. Meanwhile this transition from the given to the possible is already found in the formation of such notions as any integer or infinite line, because neither arbitrarily big numbers nor infinite lines are given in experience. But when the concept of number crystallized, then from this very concept, from the law of formation of consecutive numbers by adding one, the possibility of endless continuation of the number series was revealed. In the same way, the possibility of unlimited continuation of a line was revealed from the drawing of

³Lenin, Philosophical Notebooks

⁴Lenin, Philosophical Notebooks

lines, expressed in Euclid's second postulate: "every line can be indefinitely continued". A further process of abstraction led to the notions of the entire natural number series and the entire infinite line. In the last stage of the development of mathematics, constructing theories through a series of abstractions and the formation of concepts was qualitatively new. But as it rises in these stages of abstraction, mathematics is not at all detached from reality. It builds up new things on the basis of a reflection of reality, as a consequence of the logic of its very subject, and precisely because of this it returns to reality in its applications to problems of physics and technology. Such was the case with imaginary numbers. The same is true for other mathematical theories, however abstract they may be.

The theories of various multidimensional spaces are a characteristic example. They developed as a generalization of Euclidean geometry, in conjunction with the development of algebra and calculus, under the influence of mechanics and physics. The combination of these ideas led Riemann to construct a general theory which was further developed by other mathematicians, found a number of important applications, and finally served as the ready mathematical apparatus for Einstein's construction of the general theory of relativity, more precisely, the theory of gravitation. Abstract geometrical theories found such brilliant applications not by chance, not because of the "preordained harmony of nature and reason," but because they themselves grew on the ground of geometry which arose directly from experience, and in their emergence their creators were connected with the task of investigating real space. Riemann, in particular, directly foresaw the connection of his theory with the theory of gravitation.

Thus, in the development of mathematics the law of the cognitive process formulated by Lenin is realized: "Thinking, ascending from the concrete

to the abstract, does not depart-if it is correct ... - from the truth, but approaches it. The abstraction of matter, the law of nature, the abstraction of value, etc., in a word, all scientific (correct, serious, not nonsense) abstractions reflect nature more deeply, more correctly, more completely. From living contemplation to abstract thought and from it to practice - this is the dialectical way of cognition of truth, cognition of objective reality"⁵.

The above makes it clear that the idealistic view that mathematical theories are only contingent schemes designed to describe the data of observation or to "order the stream of sensations" on the basis of "the principle of economy of thought" is completely false.

Engels notes that the provisions of mathematics, abstracted from the real world, seem to be opposed to it and applied to its study as some ready-made schemes. Indeed, we are constantly using, for example, counting, applying it in a ready-made form. This is all the more true of theories arising at higher levels of abstraction. As an example, it has already been mentioned that Riemannian geometry served as a ready-made mathematical scheme for the theory of gravitation. But Engels explains that the possibility of such an application of mathematics to the study of the real world is based on the fact that it is borrowed from this very world and only expresses part of its inherent forms of relations and can only therefore be applied at all. The fact that many theories are constructed within mathematics does not change this. Arising as theories of possible forms of reality, they are not arbitrary because they arise necessarily out of the logic of the discipline itself and this is why they find real applications. One way or another, mathematical theories reflect reality, the only difference being that this reflection is in some cases more direct, while in others it goes through a

⁵Lenin, Philosophical Notebooks

series of abstractions, the formation of concepts, etc.

5. The last stage in the development of mathematics is characterized not only by higher stages of abstraction, it is also characterized by a substantial extension of its subject matter beyond the original understanding of quantitative relations and spatial forms.

Figures in multidimensional or infinite-dimensional spaces are certainly not spatial forms in the usual sense as we all understand them when we refer to ordinary real space rather than the abstract spaces of mathematics. These spaces have real meaning and reflect in abstract terms certain forms of reality, but these forms are only similar to spatial forms; therefore, in relation to ordinary real space they can be called "spatial-like." Speaking of multidimensional space and of figures in it, we thereby give a new content to the notion of space, so that we must clearly distinguish between the generalized abstract notion of space in mathematics, on the one hand, and the notion of space in its original sense of a universal form of matter's existence, on the other.

Another example of mathematics going beyond spatial forms and quantitative relations in the original sense of these words is the emergence at the end of the last century of a new discipline, mathematical logic, which has now reached a broad development. Its subject matter is the structure of mathematical conclusions; in other words, it studies what propositions can be deduced from given premises by given means. It investigates its subject matter, as it is peculiar to mathematics, in complete abstraction from the content, and hence it replaces propositions by formulas, and rules of inference by rules of operations with these formulas. The relations between premise and conclusion, axioms and theorems, of course, are not reduced

to spatial forms or, in the usual sense, to quantitative relations, say, to relations of volumes of concepts.

As another example, let us refer to the group theory, which can be understood as the study of symmetry in its most general form. However, a change in the symmetry of a crystal, say, when sulfur changes from rhombic to prismatic form, is a radical qualitative change in the state of matter. Thus, the theory of groups is the doctrine of such values or such definitions of objects whose change is accompanied by a radical change in the objects themselves.

Thus, the expansion of the subject matter of mathematics leads to an essential expansion of the very notion of quantitative relations and spatial forms. What, then, are the characteristic general features of this of this expanding subject of mathematics?

If we answer this question not by enumeration but by an effort to clarify what is common and typical in the subject of mathematics in all its diversity, we find the answer essentially in Engels. It suffices to take into account not only his indication of the subject of mathematics, but also the way of looking at the subject: the total abstraction of forms and relations from their content. This abstract character of mathematics also gives at the same time a definition of its subject.

The subjects of mathematics are those forms and relations of reality which objectively possess such a degree of indifference to content that they can be completely abstracted from it and defined in general terms with such clarity and precision, with such a wealth of connections preserved, as to serve as a basis for the purely logical development of theory. If such relations and forms are to be called quantitative in the general sense of the word, it may

be briefly said that mathematics is concerned with quantitative relations and forms, taken in their purest form.

Abstraction is by no means the privilege of mathematics. However, other sciences are primarily interested in the correspondence of their abstract schemes to a quite definite range of phenomena and include as one of their most important tasks the investigation of the limits of applicability to a given range of phenomena of the already established system of concepts and the corresponding change in the applied system of abstractions. Mathematics, on the contrary, by investigating general properties in complete abstraction from concrete phenomena, considers these very systems of abstractions in their abstract generality, beyond the limits of their applicability to individual concrete phenomena. We might say that mathematics is characterized by a kind of absolutization of its abstractions.

It is precisely this objective indifference to the content of the forms studied in mathematics that determines its basic features: its contemplative nature, the logical necessity and seeming inviolability of its conclusions, the emergence of new concepts and theories within it; and the same indifference to content determines the specificity of the applications of mathematics. When we have been able to translate a practical problem into the language of mathematics, we have simultaneously been able to abstract it away from the secondary concrete features of the problem and, by using general formulas and conclusions, to arrive at a definite result. The abstractness of mathematics thus constitutes its strength, and this abstractness is necessary in practice.

6 Returning now to Engels' judgment of mathematics, we see what depth and richness of content, what possibilities of development lie in this judg-

ment. Without being a mathematician himself, he gave such a profound analysis of the foundations of this science, not only because he was a brilliant thinker, but, most importantly, because he possessed dialectical materialism and was guided by it in the task of clarifying the essence of mathematics. It is therefore no wonder that no one before Engels could give such a profound and correct solution to this question. The greatest mathematicians could not solve it to such an extent.

Similarly, Lenin subsequently gave such an analysis of the problem of physics, which surpasses everything done in this field.

This proves once again the importance and power of dialectical materialism; it proves that to master science it is not enough to know its individual provisions, it is not enough even to be a creative worker in that science - it is still necessary to possess the correct general method, to possess dialectical materialism. Without this, the conclusions of science will either seem like a shapeless pile or be presented in a distorted way; instead of a correct understanding of science, a false, metaphysical, idealistic view of it will develop. Thus, for example, many mathematicians who do not master dialectical materialism are either not at all acquainted with the general questions of their science, or they interpret them quite wrongly ⁶.

At the time when Engels wrote "Anti-Dühring", i.e. in 1876-1877, non-Euclidean geometry and geometry of multidimensional spaces had just gained recognition among mathematicians, the theory of groups had just taken shape, the theory of sets was just emerging, and mathematical logic was only just beginning to emerge. It is therefore understandable that the

⁶It is interesting to note, for example, that two famous American geometers, Veblen and Whitehead, in their book "Foundations of Differential Geometry," try to define what geometry is and come to the conclusion that no such definition can be given except the following: "geometry is what specialists call geometry."

features of the new stage in the development of mathematics could not be reflected in detail by Engels; nevertheless, in his judgments we find guidance for understanding them as well.

9. Patterns of the development of mathematics

In conclusion, we will try to summarize the general patterns of the development of mathematics.

1. Mathematics is not the creation of any one historical epoch, of any one people; it is the product of a series of epochs, the product of the work of many generations. Its first notions and concepts emerged, as we have seen, in ancient times and were put into a coherent system more than two thousand years ago. In spite of all the transformations of mathematics, its concepts and conclusions persist, passing from one epoch to another, such as the rules of arithmetic or the Pythagorean theorem.

New theories incorporate previous achievements, clarifying, supplementing, and generalizing them.

At the same time, as is clear from this brief sketch of the history of mathematics above, its development is not merely an accumulation of new theorems, but includes substantial, qualitative changes. Accordingly, the development of mathematics is divided into a series of stages, the transitions between which are precisely marked by such fundamental changes in the very subject or structure of this science.

Mathematics encompasses into its scope all new areas of quantitative relations of reality. At the same time, the most important subject matter of mathematics was and remains spatial forms and quantitative relations in the simplest,

most immediate sense of these words, and mathematical comprehension of new connections and relations inevitably occurs on the basis of and in connection with the already established system of quantitative and spatial scientific notions.

Finally, the accumulation of results within mathematics itself necessarily leads both to new steps of abstraction, to new generalizing notions, and to a deeper analysis of basic and original notions.

As the oak tree in its mighty growth thickens its old branches with new layers, throws out new branches, stretches upward and deepens its roots downward, so does mathematics in its development accumulate new material in its already established fields, form new directions, ascend to new heights of abstraction and deepen in its foundations.

2. Mathematics is concerned with the real forms and relations of reality, but, as Engels said, to study these forms and relations in their pure form, it is necessary to separate them completely from their content, to leave the latter aside as something irrelevant. But there are no forms and relations outside of content; mathematical forms and relations cannot be totally indifferent to content. Hence, by its very nature, mathematics, which strives to do this separation, is striving to do the impossible. *This is the fundamental contradiction in the very nature of mathematics.* It is a specific manifestation of the general contradiction of knowledge. The reflection of every phenomenon, every aspect, every moment of reality by thought coarsens it, simplifies it, taking it out of the total relation of nature. When people, studying the properties of space, established that it has Euclidean geometry, an extremely important act of cognition was committed, but in it lay a delusion: the real properties of space were taken simplistically,

schematically, in abstraction from matter. But without this there would simply be no geometry, and it was on the basis of this abstraction (both from its internal study and from the comparison of mathematical results with new data from other sciences) that new geometrical theories were born and strengthened.

The constant resolution and restoration of this contradiction at the stages of cognition closer and closer to reality is the essence of the development of cognition. In this case, the determining factor is, of course, the positive content of cognition, the element of absolute truth in it. Cognition goes upward, not stumping in place in a simple mixture with delusion. The movement of cognition is a constant overcoming of its imprecision and limitation.

This basic contradiction entails others. We have seen this in the example of the opposites of the discrete and the continuous. (In nature there is no absolute gap between them; their separation in mathematics inevitably led to the necessity of creating new notions, more deeply reflecting reality and at the same time overcoming the inner imperfections of the existing mathematical theory.) In the same way, the contradictions of finite and infinite, abstract and concrete, form and content, etc., appear in mathematics as manifestations of its fundamental contradiction. But its decisive manifestation is that by abstracting from the concrete, by circulating in its abstract notions, mathematics is thus separated from experiment and practice; but it is only a science (i.e. has cognitive value) because it is based on practice, because it is not pure but applied mathematics. In somewhat Hegelian terms, *pure mathematics constantly "negates" itself as pure mathematics; without this it cannot have scientific meaning, cannot develop, cannot overcome the difficulties that inevitably arise within it.*

In its formal appearance, mathematical theories confront the real content as some schemes for concrete conclusions. In this case, mathematics acts as a method of formulating quantitative laws of natural science, as an apparatus for developing its theories, and as a means of solving the problems of natural science and engineering. The significance of pure mathematics at the present stage lies first of all in the mathematical method. And just as any method exists and develops not by itself, but only on the basis of its applications, in connection with the content to which it is applied, so mathematics cannot exist and develop without applications. Here again we find a unity of opposites: the general method is opposed to the concrete problem as a means of solving it, but it itself arises from the generalization of concrete material and exists, develops and finds its justification only in the solution of concrete problems.

3. Social practice plays a determining role in the development of mathematics in three ways. It poses new problems for mathematics, stimulates its development in one direction or another, and provides a criterion for the truth of its conclusions.

This can be seen most clearly in the emergence of calculus. First, it was the development of mechanics and technology that brought forward the problem of studying the dependencies of variables in their general form. Archimedes, who came close to differential and integral calculus, remained, however, within the framework of the tasks of statics, whereas in modern times it was the study of motion that gave rise to the concepts of variable and function and prompted the formalization of calculus. Newton could not develop mechanics without developing a corresponding mathematical method.

Second, it was the needs of social production that prompted the formulation and solution of all these problems. Neither ancient nor medieval society had yet had these incentives. Finally, it is crucial that mathematical analysis, at its inception, found the grounding of its conclusions in applications. This was the only reason why it could develop without the strict definitions of its main concepts (variable, function, limit) that were given later. The truth of analysis was established by applications to mechanics, physics, and engineering.

This applies to all periods in the development of mathematics. Since the seventeenth century, its development has been most directly influenced, together with mechanics, by theoretical physics and the problems of new engineering. Continuum mechanics and later field theory (heat conduction, electricity, magnetism, gravitational field) guided the development of the theory of partial differential equations. The development of molecular theory and statistical physics in general, starting from the end of the last century, served as an important stimulus for the development of probability theory, especially the theory of random processes. The theory of relativity played a crucial role in the development of Riemannian geometry with its analytic methods and generalizations.

At present, the development of new mathematical theories, like functional analysis and others, is stimulated by the problems of quantum mechanics and electrodynamics, the problems of computer science, statistical questions of physics and engineering, etc., etc. Physics and technology not only pose new challenges to mathematics, pushing it into new subjects of study, but also awaken the development of the necessary sections of mathematics that were originally formed more within itself, as was the case with Riemannian geometry. In short, the intensive development of science requires

not only that it should approach new problems, but that the necessity of solving them should be imposed by the needs of the development of society. In mathematics many theories have arisen in recent times, but only those of them are developed and firmly embedded in science which have found their applications in science and technology, or which have played the role of important generalizations of theories that have such applications. At the same time, other theories remain unmoved, such as some refined geometric theories (non-Euclidean, non-Archimedean geometries) that have not found substantial applications.

The truth of mathematical conclusions finds its final foundation not in general definitions and axioms, not in the formal rigor of the proofs, but in real applications, i.e. ultimately in practice.

In sum, the development of mathematics must be understood primarily as the result of the interaction between the logic of its subject, reflected in the internal logic of mathematics itself, the influence of production and its links with natural science. This distinction follows complex paths of struggle of opposites, including essential changes in the basic content and forms of mathematics. In content, the development of mathematics is determined by its subject, but it is driven mainly and ultimately by the needs of production. This is the basic law of mathematical development.

However, we must not forget that this is only a fundamental law and that the relation between mathematics and industry is, generally speaking, a complex one. From what has been said above, it is clear that it would be naïve to try to justify the emergence of each given mathematical theory by a direct “production order.” Moreover, mathematics, like any science, has its own relative independence, its own internal logic, reflecting, as we have

stressed, objective logic, i.e. the regularity of its subject.

4. Mathematics has always been most significantly influenced not only by social production, but by all social conditions in general. Its brilliant progress during the rise of ancient Greece, the success of algebra in Italy during the Renaissance, the development of analysis in the period following the English Revolution, the success of mathematics in France in the period adjacent to the French Revolution - all this clearly demonstrates the inseparable connection between the progress of mathematics and the general technical, cultural, and political progress of society.

This is also clearly seen in the development of mathematics in Russia. The formation of an independent Russian mathematical school, coming from Lobachevsky, Ostrogradsky, and Chebyshev, cannot be separated from the progress of Russian society as a whole. The time of Lobachevsky was the time of Pushkin, of Glinka, of the Decembrists, and the flowering of mathematics was one of the elements of the general rise.

The impact of social development in the period after the Great October Socialist Revolution, when research of fundamental importance appeared one after another with astonishing rapidity in many fields: in set theory, topology, number theory, probability theory, differential equation theory, functional analysis, algebra, geometry, was all the more convincing.

Finally, mathematics has always been and continues to be markedly influenced by ideology. As in all sciences, the objective content of mathematics is understood and interpreted by mathematicians and philosophers within the framework of one ideology or another.

In short, the objective content of science is always embedded in one ideological form or another; the unity and struggle of these dialectical opposites

- objective content and ideological forms - in mathematics, as in all science, play a far-reaching role in its development.

The struggle between 'materialism, which corresponds to the objective content of science, and idealism, which contradicts this content and perverts its understanding, runs through the entire history of mathematics. This struggle is clearly marked already in ancient Greece, where the idealism of Pythagoras, Socrates and Plato opposed the materialism of Thales, Democritus and other philosophers who created Greek mathematics. With the development of the slave system, the upper class of society was cut off from participation in production, considering it the destiny of the lower class, and this gave rise to the separation of "pure" science from practice. Only purely theoretical geometry was considered worthy of the attention of a true philosopher. Characteristically, Plato considered the emerging research on some mechanical curves and even conic sections to remain outside the scope of geometry, as they "do not bring us into communion with eternal and incorporeal ideas" and "need to apply the tools of vulgar craftsmanship".

A vivid example of the struggle of materialism against idealism in mathematics is represented by the activity of Lobachevsky, who put forward and defended a materialistic understanding of mathematics against the idealistic views of Kantianism.

The Russian mathematical school in general is characterized by a materialistic tradition. Thus, Chebyshev explicitly stressed a decisive importance of practice, while Lyapunov expressed the style of the Russian mathematical school in the following remarkable words: "Detailed elaboration of questions, especially important from the point of view of application and at the

same time presenting special theoretical difficulties, requiring invention of new methods and ascending to the principles of science, then generalization of the obtained conclusions and creation by this way of more or less general theory". Generalizations and abstractions not in themselves but in connection with concrete material, theorems and theories not in themselves but in the general connection of science, leading ultimately to practice, are what actually turns out to be important and promising⁷.

Such were the aspirations of such great scientists as Gauss and Riemann.

However, with the development of capitalism in Europe, materialistic views, which reflected the advanced ideology of the rising bourgeoisie of the 16th and early 19th centuries, began to be replaced by idealistic views. For example, Cantor (1846-1918), creating the theory of infinite sets, directly referenced God, arguing in the spirit that the infinite sets have an absolute existence in the divine mind. Poincaré, the greatest French mathematician of the late nineteenth and early twentieth centuries, put forward an idealistic concept of "conventionalism," according to which mathematics is a scheme of conditional conventions adopted for the convenience of describing the variety of experience. Thus, according to Poincaré, the axioms of Euclidean geometry are nothing more than conditional agreements, and their meaning is determined by convenience and simplicity, but not by correspondence to the real reality. That is why Poincaré said that, for example, in physics one would rather give up the law of rectilinear propagation of light than Euclidean geometry. This point of view was refuted by the development of the theory of relativity, which, contrary to all the "simplic-

⁷In general, an understanding of the necessary connections of individual areas of mathematics with each other, with natural science and practice is extremely important not only for the correct view of mathematics itself, but also for guiding scientists in choosing directions and subjects of research.

ity" and "convenience" of Euclidean geometry, in full agreement with the materialistic ideas of Lobachevsky and Riemann, led to the conclusion that the real geometry of space is different from Euclidean.

Because of the difficulties encountered in set theory and the need to analyze the basic concepts of mathematics, different currents emerged among mathematicians in the early twentieth century. The unity in understanding the content of mathematics was lost; different mathematicians began to view not only the general foundations of science in different ways, as had been the case before, but even the meaning and significance of particular concrete results and proofs began to be evaluated differently. Conclusions that seemed meaningful and substantial to some, others declared devoid of meaning and significance. Idealistic currents of "logicism," "intuitionism," "formalism," etc. emerged.

Logicians claim that all mathematics is derivable from concepts of logic. Intuitionists see the source of mathematics in intuition and give meaning only to what is intuitively perceived. This is why, in particular, they deny the significance of Cantor's theory of infinite sets at all. Moreover, intuitionists deny the simple meaning even of statements such as the theorem that every algebraic equation of degree n has n roots. For them, the statement is empty as long as no way of calculating the roots is specified. Thus, the complete denial of the objective meaning of mathematics led intuitionists to denigrate, as "devoid of meaning," much of the achievements of mathematics. The most extreme of them went so far as to claim that there are as many mathematics as there are mathematicians.

An attempt in his own way to save mathematics from this kind of attack was made by the greatest mathematician of the beginning of our century,

D. Hilbert. The essence of his idea was to reduce mathematical theories to purely formal operations on symbols according to prescribed rules. The calculation was that such a perfectly formal approach would eliminate all difficulties, because the subject of mathematics would be symbols and the rules of dealing with them without any relation to their meaning. This is the installation of formalism in mathematics. According to the intuitionist Brauer, for the formalist, the truth of mathematics is on paper, whereas for the intuitionist it is in the mind of the mathematician.

It is easy to see, however, that they are both wrong, for mathematics, and with it what is written on paper and what the mathematician thinks, reflects reality, and the truth of mathematics lies in its correspondence with objective reality. By detaching mathematics from material reality, all these currents turn out to be idealistic.

Hilbert's idea was defeated as a result of its own development. The Austrian mathematician Gödel proved that even arithmetic could not be formalized completely as Hilbert had hoped. Gödel's conclusion clearly revealed the inner dialectic of mathematics, which does not allow any of its fields to be exhausted by a formal calculus. Even the simplest infinity of a natural series of numbers proved to be an inexhaustible finite scheme of symbols and rules for dealing with them. Thus, it has been mathematically proved what was expressed in general terms by Engels:

"Infinity is contradiction. The destruction of this contradiction would be the end of infinity"⁸! Hilbert expected to enclose mathematical infinity within finite schemes and thus eliminate all contradictions and difficulties. This proved impossible.

⁸Engels, *Anti-Dühring*

But under capitalism, conventionalism, intuitionism, formalism, and other such currents not only persist but are supplemented by new variants of idealistic views of mathematics. Theories related to the logical analysis of the foundations of mathematics are used substantially in some new variants of subjective idealism. Subjective idealism now uses mathematics, in particular mathematical logic, as much as it does physics, and hence questions of understanding the foundations of mathematics become particularly acute.

Thus, under capitalism, the difficulties in the development of mathematics have given rise to an ideological crisis of this science, similar in its fundamentals to the crisis of physics, the essence of which was clarified by Lenin in his brilliant work "Materialism and Empirio-criticism." This crisis does not mean that mathematics in capitalist countries is completely arrested in its development. A number of scientists, standing on clearly idealistic positions, are making important, sometimes outstanding advances in solving specific mathematical questions and developing new theories. Suffice it to mention the brilliant development of mathematical logic.

The fundamental flaw of the widespread capitalist view of mathematics is its idealism and metaphysics: disconnecting mathematics from reality and neglecting the real development of mathematics. Logic, intuitionism, formalism, and other similar schools of thought emphasize only one aspect of mathematics, its relations to logic, intuitive clarity, formal rigor, etc. They tend to groundlessly exaggerate, to absolutize its importance, to detach it from reality, losing sight of mathematics as a whole by focusing on this one aspect of mathematics per se. It is precisely because of this one-sidedness that none of these currents, for all the finesse and depth of their individual conclusions, can lead to a correct understanding of mathematics. In contrast to the various currents and shades of idealism and metaphysics,

dialectical materialism views mathematics, like science as a whole, as it is, in all the richness and complexity of its connections and development. And precisely because dialectical materialism seeks to understand the richness and complexity of science's relations to reality, the complexity of its development from the simple generalization of experience to higher abstractions and from these to practice, precisely because it constantly brings its approach to science into accord with its objective content, with its new discoveries, precisely because, and for this reason alone, it is the only truly scientific philosophy that leads to a correct understanding of science in general and, in particular, of mathematics.

